

MODULAR REPRESENTATIONS OF REDUCTIVE GROUPS AND GEOMETRY OF AFFINE GRASSMANNIANS

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ABSTRACT. By the geometric Satake isomorphism of Mirković and Vilonen, decomposition numbers for reductive groups can be interpreted as decomposition numbers for equivariant perverse sheaves on the complex affine Grassmannian of the Langlands dual group. Using a description of the minimal degenerations of the affine Grassmannian obtained by Malkin, Ostrik and Vybornov, we are able to recover geometrically some decomposition numbers for reductive groups. In the other direction, we can use some decomposition numbers for reductive groups to prove geometric results, such as a new proof of non-smoothness results, and a proof that some singularities are not equivalent (a conjecture of Malkin, Ostrik and Vybornov). We also give counterexamples to a conjecture of Mirković and Vilonen stating that the stalks of standard perverse sheaves over the integers on the affine Grassmannian are torsion-free, and propose a modified conjecture, excluding bad primes.

INTRODUCTION

In [Jut08], we introduced decomposition numbers for perverse sheaves and calculated them for simple and minimal singularities. This has applications in the modular representation theory of Weyl groups, using a modular Springer correspondence [Jut07b].

In this article, we will use another bridge between representation theory and geometry, provided by the geometric Satake isomorphism of Mirković and Vilonen [MV07]. It is an equivalence of tensor categories between the category of representations of a reductive algebraic group G over an arbitrary commutative ring \mathbb{E} , and a category of equivariant perverse sheaves on the complex affine Grassmannian of the Langlands dual group, with \mathbb{E} coefficients.

In the first part of this article, we give applications from geometry to the representation theory of reductive algebraic group schemes, using results of Malkin, Ostrik and Vybornov describing the minimal degenerations in the affine Grassmannian [MOV05]. In Section 1, we review the results of [MV07]. We deduce that decomposition numbers for reductive groups (that is, the multiplicities of simple modules in Weyl modules) are decomposition numbers for perverse sheaves. In Section 2, we review the results of [MOV05]. It turns out that most minimal degenerations in the affine Grassmannian are either simple or minimal singularities, which we have studied in [Jut08]. So we can recover decomposition numbers for reductive groups in this way. First, we remark in Section 3 that a Levi lemma known in representation theory [Jan03, §5.21 (2) p. 230] can be interpreted geometrically by a Levi lemma used by Malkin, Ostrik and Vybornov. Then, in Section 4, for the case of a simple singularity we recover the fact that for two dominant weights which are adjacent in the order of all dominant weights, the decomposition number

is one if there is a wall between them, and zero otherwise [Jan03, Cor. 6.24 p.249]. In Section 5, for the case of a minimal singularity, we recover the multiplicity of the trivial module in the Weyl module of highest weight the dominant short root, which was computed in [CPS75, Theorem 1.1]. Finally, in Section 6, we remark that the calculations we did in [Jut07a] for minimal singularities, together with the results in [Jut08], provide counterexamples to a conjecture in [MV07] stating that the standard \mathbb{Z}_ℓ -perverse sheaves on the affine Grassmannian should have torsion-free stalks. We propose to modify the conjecture by requiring that ℓ be good.

In the second part of this article, we go in the other direction and use some decomposition numbers for reductive groups to prove geometric results. In Section 7, we compute the decomposition number for $G = \mathrm{Spin}_{2n+1}$ corresponding for the weights $\lambda = \varpi_1 + \varpi_n$ and $\mu = \varpi_n$ (in the numbering of [Bou68]), corresponding to the quasi-minimal singularity of type ac_n . In Section 8, we prove non-equivalences of singularities conjectured in [MOV05]. The argument is as follows: if the singularities were equivalent, then the stalks of the intersection cohomology complexes would be the same, both in characteristic zero and in characteristic ℓ , so the decomposition numbers for perverse sheaves would be the same, and thus also the decomposition numbers for reductive groups. We then see that there is always a prime number ℓ for which these decomposition numbers are different. Finally, in Section 9, we give a representation-theoretic proof of the fact that the smooth locus of an orbit on the affine Grassmannian is reduced to that orbit (this result was already proved, in different ways, in [EM99] and [MOV05]). For each minimal degeneration, we give a prime number ℓ for which the corresponding decomposition number is non-trivial, which implies that it is not \mathbb{F}_ℓ -smooth, and thus not smooth.

The methods and results in this work show again that it is useful to consider perverse sheaves with coefficients of arbitrary characteristic. Considering the intersection cohomology complexes modulo all possible primes ℓ , we get a finer invariant than just the characteristic zero case. In this way, one can prove non-trivial geometric results such as non-smoothness or non-equivalences, which cannot be seen in the rational intersection cohomology stalks. In the other direction, a deep understanding of the geometry of the affine Grassmannians, including the determination of the stalks of the intersection cohomology complexes in positive characteristic, would give the modular characters of the reductive groups.

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Part 1. From geometry to representation theory

1. GEOMETRIC SATAKE ISOMORPHISM

By the geometric Satake isomorphism of Mirković and Vilonen [MV07], the category of representations of a connected reductive algebraic group scheme over any commutative ring \mathbb{E} is naturally equivalent to a certain category of equivariant perverse sheaves on the complex affine Grassmannian of the Langlands dual group G^\vee , with \mathbb{E} coefficients.

Let G be a split connected reductive algebraic group scheme defined over \mathbb{Z} . We fix a split maximal torus T in G . Then we have the character lattice $X(T)$ and the cocharacter lattice $Y(T)$, with the canonical perfect pairing $\langle -, - \rangle : X(T) \times Y(T) \rightarrow \mathbb{Z}$, and the root systems Φ in $X(T)$ and Φ^\vee in $Y(T)$. We also fix a Borel subgroup B in G containing T , corresponding to some choice of subset of positive roots Φ^+ in Φ , defined by a basis Δ . We denote by $X(T)^+$ the set of dominant weights.

Then the irreducible $G_{\mathbb{C}}$ -modules are the Weyl modules $V_{\mathbb{C}}(\lambda)$, for $\lambda \in X(T)^+$. Let ℓ be a prime number. There is a procedure of reduction modulo ℓ . The choice of a highest weight vector v_+ in $V_{\mathbb{C}}(\lambda)$ (which is unique up to a non-zero scalar) determines an integral Weyl module $V_{\mathbb{Z}}(\lambda)$ (it is a module for $\mathcal{U}_{\mathbb{Z}}$, Kostant's \mathbb{Z} -form of the enveloping algebra \mathcal{U} of the Lie algebra \mathfrak{g} of G). Let $\mathbb{F} = \overline{\mathbb{F}}_{\ell}$. Then the Weyl module $V(\lambda)$ for $G_{\mathbb{F}}$ can be defined as $\mathbb{F} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)$. It turns out that this is the universal highest weight module of highest weight λ for $G_{\mathbb{F}}$. It has a unique simple quotient $L(\lambda)$. The $L(\lambda)$, for $\lambda \in X(T)^+$, form a full set of representatives of isomorphism classes of simple modules for $G_{\mathbb{F}}$. We also have the induced modules $H^0(\lambda)$, for $\lambda \in X(T)^+$. There is a natural morphism $V(\lambda) \rightarrow H^0(\lambda)$, and $L(\lambda)$ can also be seen as the image of this morphism. We denote by $G_{\mathbb{F}}\text{-mod}$ the category of rational representations of $G_{\mathbb{F}}$.

To use a uniform notation, if \mathbb{E} is any commutative ring, we will denote by $V_{\mathbb{E}}(\lambda)$ and $H_{\mathbb{E}}^0(\lambda)$ the Weyl and induced modules for $G_{\mathbb{E}}$, defined similarly, and if \mathbb{E} is a field, we will write $L_{\mathbb{E}}(\lambda)$ for the simple modules. We want to let \mathbb{E} vary in an ℓ -modular system $(\mathbb{K}, \mathbb{O}, \mathbb{F})$, both for the representation theory and for the coefficients of the perverse sheaves. That is, we can take for \mathbb{O} a discrete valuation ring with quotient field \mathbb{K} and of characteristic zero and residue field \mathbb{F} of characteristic ℓ , for example finite extensions of \mathbb{Q}_{ℓ} , \mathbb{Z}_{ℓ} and \mathbb{F}_{ℓ} , as in [Jut08]. But here, since we work in the complex topology, we can take arbitrary commutative rings as coefficients, such as \mathbb{C} , \mathbb{Z} and $\overline{\mathbb{F}}_{\ell}$.

The character of the Weyl module $V(\lambda)$ is the same as its counterpart over \mathbb{C} . It is given by Weyl's character formula. The characters of the simple modules $L(\mu)$ are not known in general. Their determination is equivalent to the determination of the multiplicities $d_{\lambda\mu}^G := [V(\lambda) : L(\mu)]$. Obviously, we have $d_{\lambda\lambda}^G = 1$, and $d_{\lambda\mu}^G \neq 0$ implies $\mu \leq \lambda$ (for the usual order on $X(T)$, that is, $\lambda - \mu$ has to be a non-negative linear combination of positive roots).

Now let $G_{\mathbb{C}}^{\vee}$ denote the Langlands dual group, defined over the complex numbers. Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. We denote by $\mathcal{G}r$ the affine Grassmannian $G^{\vee}(\mathcal{K})/G^{\vee}(\mathcal{O})$. It is an ind-variety (a direct limit of varieties). If \mathbb{E} is any commutative ring, let $\text{Perv}_{G_{\mathcal{O}}^{\vee}}(\mathcal{G}r, \mathbb{E})$ denote the category of $G_{\mathcal{O}}^{\vee}$ -equivariant perverse sheaves on $\mathcal{G}r$ with \mathbb{E} coefficients. The $G_{\mathcal{O}}^{\vee}$ -orbits on $\mathcal{G}r$ are parametrized by $X(T)^+$. We denote the orbit corresponding to λ by $\mathcal{G}r_{\lambda}$. It is of dimension $d_{\lambda} := 2\langle \lambda, \rho^{\vee} \rangle$, where ρ^{\vee} is the sum of the fundamental coweights. For λ in $X(T)^+$, we set ${}^p\mathcal{I}_!(\lambda, \mathbb{E}) = {}^pj_{\lambda!}(\overline{\mathcal{G}r}_{\lambda}, \mathbb{E}[d_{\lambda}])$, where j_{λ} is the inclusion of $\mathcal{G}r_{\lambda}$ in $\mathcal{G}r$ and p is the middle perversity, and similarly for ${}^p\mathcal{I}_*(\lambda, \mathbb{E})$ and ${}^p\mathcal{J}_*(\lambda, \mathbb{E})$. If $\mathbb{E} = \mathbb{O}$, we can consider these three extensions relative to the dual perversity p_+ instead of p [BBD82, Jut08].

Mirkovic and Vilonen [MV07] related the representation theory of $G_{\mathbb{E}}$ with \mathbb{E} -perverse sheaves on the affine Grassmannian of G^{\vee} . More precisely, their main result is as follows:

Theorem 1.1 (Mirkovic-Vilonen). *We have an equivalence of tensor categories*

$$\Xi : G_{\mathbb{E}}\text{-mod} \xrightarrow{\sim} \text{Perv}_{G_{\mathcal{O}}^{\vee}}(\mathcal{G}r, \mathbb{E})$$

which sends the natural morphism $V_{\mathbb{E}}(\lambda) \rightarrow H_{\mathbb{E}}^0(\lambda)$ to the natural morphism $\mathcal{I}_!(\lambda, \mathbb{E}) \rightarrow \mathcal{J}_*(\lambda, \mathbb{E})$. Thus, if \mathbb{E} is a field, Ξ sends the simple $L_{\mathbb{E}}(\lambda)$ to the simple $\mathcal{I}_*(\lambda, \mathbb{E})$.

Moreover, this equivalence is compatible with the extension of scalars $\mathbb{K} \otimes_{\mathbb{O}} -$ and the modular reduction $\mathbb{F} \otimes_{\mathbb{O}} -$ (if we take a torsion-free object in either category for $\mathbb{E} = \mathbb{O}$, its modular reduction is an object of the corresponding category for $\mathbb{E} = \mathbb{F}$).

They also prove that ${}^p\mathcal{I}_!(\lambda, \mathbb{O}) \simeq {}^p\mathcal{I}_*(\lambda, \mathbb{O})$. This implies

$${}^p\mathcal{I}_!(\lambda, \mathbb{O}) \simeq {}^{p+}\mathcal{I}_!(\lambda, \mathbb{O}) \simeq {}^p\mathcal{I}_*(\lambda, \mathbb{O})$$

and

$${}^{p+}\mathcal{I}_*(\lambda, \mathbb{O}) \simeq {}^p\mathcal{J}_*(\lambda, \mathbb{O}) \simeq {}^{p+}\mathcal{J}_*(\lambda, \mathbb{O})$$

and also

$$\mathbb{F}^p\mathcal{I}_!(\lambda, \mathbb{O}) \simeq {}^p\mathcal{I}_!(\lambda, \mathbb{F}) \simeq \mathbb{F}^{p+}\mathcal{I}_!(\lambda, \mathbb{O}) \simeq \mathbb{F}^p\mathcal{I}_*(\lambda, \mathbb{O})$$

and

$$\mathbb{F}^{p+}\mathcal{I}_*(\lambda, \mathbb{O}) \simeq \mathbb{F}^p\mathcal{J}_*(\lambda, \mathbb{O}) \simeq {}^p\mathcal{J}_*(\lambda, \mathbb{F}) \simeq \mathbb{F}^{p+}\mathcal{J}_*(\lambda, \mathbb{O})$$

Here, we denote simply by $\mathbb{F}(-)$ the functor of modular reduction. See [Jut08] for general results of this kind.

We can consider the decomposition numbers for $G_{\mathcal{O}}^{\vee}$ -equivariant perverse sheaves on $\mathcal{G}r$, defined by

$$d_{\lambda\mu}^{\mathcal{G}r} := [\mathbb{F}^p\mathcal{I}_*(\lambda, \mathbb{O}) : {}^p\mathcal{I}_*(\mu, \mathbb{F})] = [{}^p\mathcal{I}_!(\lambda, \mathbb{F}) : {}^p\mathcal{I}_*(\mu, \mathbb{F})]$$

Let us give an immediate consequence of the result of Mirković and Vilonen in terms of decomposition numbers:

Corollary 1.2. *For λ, μ in $X(T)^+$, we have*

$$d_{\lambda\mu}^G = d_{\lambda\mu}^{\mathcal{G}r}$$

Therefore, this decomposition number will just be denoted by $d_{\lambda\mu}$.

2. MINIMAL DEGENERATIONS

In [Jut08], we introduced decomposition numbers for perverse sheaves, described some of their properties, and computed them for simple surface singularities and for minimal singularities. This had applications for the modular representation theory of Weyl groups, using a modular Springer correspondence making a link with modular perverse sheaves on the nilpotent cone: we showed in [Jut07b] that decomposition numbers for Weyl groups are particular cases of decomposition numbers for perverse sheaves on the nilpotent cone.

It turns out that simple and minimal singularities also occur in the affine Grassmannian, and in fact most minimal degenerations in $\mathcal{G}r$ are of either kind, by the work of Malkin, Ostrik and Vybornov [MOV05]. In non-simply-laced types, a few others show up, which they call quasi-minimal.

A minimal degeneration in a stratified space is a pair of strata which are adjacent in the order given by the inclusion of closures. In the affine Grassmannian $\mathcal{G}r$, these are parametrized by pairs of adjacent dominant weights (λ, μ) (in the usual order), *i.e.* such that $\lambda > \mu$ and there is no dominant weight ν with $\lambda > \nu > \mu$. Such a pair is also called a minimal degeneration, and is denoted by $\lambda \rightsquigarrow \mu$. They were classified by Stembridge [Ste98].

For β in Q , we denote by $\text{supp}(\beta)$ the Dynkin subdiagram consisting in the simple roots appearing in the decomposition of β with a non-zero coefficient. For a minimal degeneration $\lambda \rightsquigarrow \mu$, $\text{supp}(\lambda - \mu)$ is clearly connected. For $I \subset \Delta$ and $\lambda = \sum_{\alpha \in \Delta} \lambda_\alpha \varpi_\alpha$ a weight, we set $\lambda_I = \sum_{\alpha \in I} \lambda_\alpha \varpi_\alpha$.

Theorem 2.1 (Stembridge). *Let $\lambda > \mu$ be a pair of weights. We set $\beta = \lambda - \mu$ and $I = \text{supp}(\beta) \subset \Delta$. Then we have $\lambda \rightsquigarrow \mu$ if and only if one of the following holds:*

- Case (1) β is a simple root.*
- Case (2) β is the short dominant root of Φ_I and $\langle \mu, \alpha^\vee \rangle = 0$ for all α in I .*
- Case (3) β is the short dominant root of Φ_I , Φ_I is of type B_n , and $\mu_I = \varpi_\alpha$, where α is the unique short simple root in I .*
- Case (4) Φ_I is of type G_2 , $\lambda_I = \varpi_{\alpha_1} + \varpi_{\alpha_2}$, $\mu_I = 2\varpi_{\alpha_1}$, where $I = \{\alpha_1, \alpha_2\}$ with α_1 short and α_2 long.*
- Case (5) Φ_I is of type G_2 , $\lambda_I = \varpi_{\alpha_2}$, $\mu_I = \varpi_{\alpha_1}$, where $I = \{\alpha_1, \alpha_2\}$ as in the previous case.*

Note that, in the two last cases, we have $\beta = \varpi_{\alpha_2} - \varpi_{\alpha_1} = \alpha_1 + \alpha_2$. In this situation, if we assume Φ to be irreducible, then $\Phi = \Phi_I$.

Now we come to the description of the minimal degenerations. For the notion of smooth equivalence of singularities and the notation Sing , we refer to [KP81].

Theorem 2.2 (Malkin-Ostrik-Vybornov). *Let $\lambda \rightsquigarrow \mu$ be a minimal degeneration. Let $\beta = \lambda - \mu$ and $I = \text{supp}(\beta)$. We set $\lambda = \sum_{\alpha \in \Delta} \lambda_\alpha \varpi_\alpha$ and $\mu = \sum_{\alpha \in \Delta} \mu_\alpha \varpi_\alpha$.*

- Case (1) If β is a simple root, then $\text{Sing}(\overline{\mathcal{G}r}_\lambda, \mathcal{G}r_\mu)$ is a Kleinian singularity of type $A_{\lambda_\beta - 1} = A_{\mu_\beta + 1}$.*

Case (2) If β is the short dominant root of Φ_I and $\mu_I = 0$, then $\text{Sing}(\overline{\mathcal{G}r}_\lambda, \mathcal{G}r_\mu)$ is a minimal singularity of the type of Φ_I^\vee , the root subsystem of Φ^\vee generated by the α^\vee , $\alpha \in I$.

So most minimal degenerations are Kleinian or minimal singularities, and we will be able to apply our study of decomposition numbers for perverse sheaves on these singularities in [Jut08]. Malkin, Ostrik and Vybornov called the remaining minimal degenerations *quasi-minimal singularities*. They are denoted by ac_n in case (3), ag_2 in case (4), and cg_2 in case (5).

The minimal degenerations can be studied using a transverse slice $S(\lambda, \mu) = \overline{\mathcal{G}r}_\lambda \cap L^{<0}G.\mu$, with the notation of [MOV05]. We denote by $d(\lambda, \mu)$ the dimension of $S(\lambda, \mu)$. We set:

$$m_\mu(\lambda, q) = \sum_{i \geq 0} \dim \mathcal{H}_\mu^{i-d(\lambda, \mu)} {}^p\mathcal{J}_{!*}(\overline{\mathcal{G}r}_\lambda, \mathbb{K}).q^i$$

Proposition 2.3. *The codimension $d = d(\lambda, \mu)$ and intersection cohomology invariants $m_\mu(\lambda, q)$ over \mathbb{K} of the minimal degenerations are given by:*

Case (1) $d = 2$ and $m_\mu(\lambda, q) = 1$;

Case (2) $d = 2h^\vee(G^\vee) - 2$, where $h^\vee(G^\vee)$ is the dual Coxeter number of G^\vee , and $m_\mu(\lambda, q) = \sum_{i=1}^t q^{e_i-1}$, where t is the number of long simple roots in Φ_I , and the e_i are the exponents of W_I ;

Case (3) $d = 2n$ and $m_\mu(\lambda, q) = \sum_{i=0}^{n-1} q^i$, as for the minimal singularity a_n ;

Case (4) $d = 4$ and $m_\mu(\lambda, q) = 1 + q$, as for the minimal singularity a_2 ;

Case (5) $d = 4$ and $m_\mu(\lambda, q) = 1$, as for the minimal singularity c_2 .

Note that, for simple and minimal singularities, we computed the local intersection cohomology over the integers in [Jut08]. For minimal singularities, this uses the computation of the integral cohomology of the minimal orbit in [Jut07a].

Malkin, Ostrik and Vybornov were able to prove the following result in a way completely different from Evens and Mirković [EM99]:

Theorem 2.4. *The smooth locus of $\overline{\mathcal{G}r}_\lambda$ is just $\mathcal{G}r_\lambda$.*

They argue as follows. It is enough to check that $\overline{\mathcal{G}r}_\lambda$ is singular along every irreducible component of the boundary $\overline{\mathcal{G}r}_\lambda - \mathcal{G}r_\lambda$, which are precisely the Schubert varieties $\overline{\mathcal{G}r}_\mu$ for all minimal degenerations $\lambda \rightsquigarrow \mu$. So they have to check that all minimal degenerations are singular, and they can use their classification. Simple singularities and minimal singularities are known to be singular; for ac_n and ag_2 singularities, they computed the rational intersection cohomology and found that they are not rationally smooth, hence they are not smooth; finally, for the cg_2 singularity, which is rationally smooth, they use Kumar's criterion. The equivariant multiplicity is the integer 27 divided by a product of weights, which indicates that it is rationally smooth, but not smooth (otherwise, the numerator would be 1). We will provide a representation-theoretic proof of the theorem in Section 9.

They also conjecture that the singularities a_2 , ac_2 and ag_2 (resp. c_2 and cg_2) are pairwise non-equivalent. We will prove this conjecture in Section 8.

3. A LEVI LEMMA

Malkin, Ostrik and Vybornov use the following geometric Levi lemma [MOV05, §3]:

Lemma 3.1. *Let $I \subset \Delta$. If $\lambda - \mu \in \mathbb{N}I$, then we have (with obvious notations):*

$$\text{Sing}(\overline{\mathcal{G}r}_\lambda, \mathcal{G}r_\mu) = \text{Sing}(\overline{\mathcal{G}r}_{\lambda_I}^{L_I^\vee}, \mathcal{G}r_{\mu_I}^{L_I^\vee})$$

If λ and μ are as in the lemma, then any pair of weights $\nu \geq \zeta$ in the interval $[\mu, \lambda]$ also satisfies this property, so the lemma can be applied for all of them. Thus, we have the same intersection cohomology stalks in both situations (for G or L_I), either with ordinary or modular coefficients, for all the interval. Since we only consider constant local systems, this implies that the decomposition numbers for perverse sheaves are the same for all the interval. By the geometric Satake isomorphism, the same is true on the representation theoretic side. So we recover a Levi lemma which was already known in representation theory [Jan03, §5.21 (2) p. 230]:

Corollary 3.2. *If $I \subset \Delta$ and $\lambda - \mu \in \mathbb{Z}I$, then we have (with obvious notations):*

$$[V(\lambda) : L(\mu)] = [V_I(\lambda) : L_I(\mu)]$$

4. SIMPLE SINGULARITIES

Theorem 4.1. *In case (1), we have $d_{\lambda\mu} = 1$ if ℓ divides λ_β , and 0 otherwise.*

Proof. The decomposition number for a simple singularity is given in [Jut08, §4.3].

□

Thus we recover geometrically a result that can be found in [Jan03, Cor. 6.24 p.249].

5. MINIMAL SINGULARITIES

Theorem 5.1. *In case (2), we have $d_{\lambda\mu} = \dim_{\mathbb{F}_\ell} \mathbb{F}_\ell \otimes_{\mathbb{Z}} P(\Phi_{I_{\text{sh}}})/Q(\Phi_{I_{\text{sh}}})$, where I_{sh} is the set of roots of minimal length in I , for any W_I -invariant scalar product.*

Proof. We have a minimal singularity of the type of Φ_I^\vee . By [Jut07a] and [Jut08, §5], the decomposition number modulo ℓ is given by this formula: the long simple coroots correspond to the short simple roots. □

Thus we recover a result of Cline, Parshall and Scott [CPS75, Theorem 1.1]. Their formulation is slightly different (they use the rank of the Cartan matrix of $\Phi_{I_{\text{sh}}}$), but it is easily seen to be equivalent (the Cartan matrix of a root system is the matrix of the inclusion of the coroot lattice into the coweight lattice, for some choice of basis of simple coroots, determining the basis of fundamental coweights; moreover, the finite abelian groups P/Q and P^\vee/Q^\vee are in duality).

6. ON THE TORSION OF THE STALKS

Mirković and Vilonen conjectured [MV07, Conjecture 13.3] that the stalks of the standard sheaves ${}^p\mathcal{J}_!(\lambda, \mathbb{Z}_\ell) = {}^p\mathcal{J}_{!*}(\lambda, \mathbb{Z}_\ell)$ are torsion-free (they actually state this conjecture for \mathbb{Z} coefficients, but it is equivalent to this property holding for \mathbb{Z}_ℓ coefficients for all prime numbers ℓ).

But we explained in [Jut08] that the stalk at 0 of the intersection cohomology complex of a minimal singularity is given by the first half of the cohomology of the minimal nilpotent orbit, and by the calculations in [Jut07a, §3], we see that there is ℓ -torsion for $\ell = 2$ in types B_n, C_n, D_n, F_4 for $\ell = 2, 3$ in types E_6, E_7 , for $\ell = 2, 3, 5$ in type E_8 , and for $\ell = 3$ in type G_2 . However, we note that for a minimal singularity, all the primes that appear are bad primes for G^\vee (recall

that we consider the affine Grassmannian for G^\vee). Therefore, we may propose the following modified conjecture:

Conjecture 6.1. *If ℓ is good for G^\vee , then the stalks of the standard perverse sheaf $\mathcal{J}_!(\lambda, \mathbb{Z}_\ell) = \mathcal{J}_{!*}(\lambda, \mathbb{Z}_\ell)$ are torsion-free.*

Part 2. From representation theory to geometry

7. A DECOMPOSITION NUMBER FOR $G = \text{Spin}_{2n+1}$

In [MOV05], it is conjectured that the quasi-minimal singularities are not equivalent to minimal singularities. We will be able to prove these non-equivalences, using decomposition numbers for reductive groups. In order to deal with the ac_n singularity, we will determine the corresponding decomposition number for $G = \text{Spin}_{2n+1}$, the simply-connected simple group of type B_n (we have not found it in the literature). For the proof that the singularities a_n and ac_n are not equivalent, we actually only need a much weaker statement, which gives a necessary condition for the decomposition number to be non-zero, and which follows from the strong linkage principle (see remark 8.2). But for the representation theoretic proof of non-smoothness, we need a non-vanishing result. Besides, along the way we will calculate a bilinear form which is likely to be interpreted geometrically. Let us mention that the tables in [Lüb01] were very useful to conjecture the result of this section.

We will use the notation of Bourbaki for the root system Φ of type B_n . So we view Φ in $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$, and the simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq n-1$, and $\alpha_n = \varepsilon_n$. We denote by W the Weyl group, and by $(\cdot|\cdot)$ the W -invariant perfect pairing for which $(\varepsilon_1, \dots, \varepsilon_n)$ is orthonormal. We have $\rho = \varpi_1 + \dots + \varpi_n = \frac{1}{2}((2n-1)\varepsilon_1 + (2n-3)\varepsilon_2 + \dots + \varepsilon_n)$.

The decomposition number we are interested is $d_{\lambda\mu}$, where $\lambda = \varpi_1 + \varpi_n = \frac{1}{2}(3\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n)$, and $\mu = \varpi_n = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n)$. We note that $\lambda - \mu = \varpi_1 = \varepsilon_1 = \alpha_1 + \dots + \alpha_n$, and $\lambda \rightsquigarrow \mu$ is a minimal degeneration.

Lemma 7.1. *We have $|W.\lambda| = n.2^n$ and $|W.\mu| = 2^n$.*

Proof. Since λ is dominant, we have $W_\lambda = W_{I_\lambda}$, where $I_\lambda = \{\alpha_2, \dots, \alpha_{n-1}\}$ is the set of simple roots orthogonal to λ . It is a Weyl group of type A_{n-2} . So we have

$$|W.\lambda| = |W : W_\lambda| = \frac{2^n n!}{(n-1)!} = n.2^n.$$

Similarly, we have $W_\mu = W_{I_\mu}$, with $I_\mu = \{\alpha_1, \dots, \alpha_{n-1}\}$. It is a Weyl group of type A_{n-1} . So we have

$$|W.\mu| = |W : W_\mu| = \frac{2^n n!}{n!} = 2^n.$$

□

Lemma 7.2. *The characters of the Weyl modules $V(\lambda)$ and $V(\mu)$ are given by*

$$\chi(\lambda) = \sum_{w \in W/W_\lambda} e(w.\lambda) + n \sum_{w \in W/W_\mu} e(w.\mu)$$

and

$$\chi(\mu) = \sum_{w \in W/W_\mu} e(w.\mu).$$

Proof. The result for $\chi(\mu)$ is clear since μ is a minuscule weight.

The only dominant weight below λ (in the usual order) is μ . By W -invariance, the multiplicity of the weights in the orbit of λ is one, and so we only have to determine the multiplicity of the weights in the orbit of μ . For this, we can use Freudenthal's formula [Bou68, Chap. 8, §9, ex. 5], which in our case is more convenient than Weyl's formula.

$$\begin{aligned}
& ((\lambda + \rho \mid \lambda + \rho) - (\mu + \rho \mid \mu + \rho)) \dim V(\lambda)_\mu \\
&= 2 \sum_{\alpha \in \Phi^+} \sum_{m \geq 1} (\mu + m\alpha \mid \alpha) \dim V(\lambda)_{\mu+m\alpha} \\
&= 2 \sum_{1 \leq i \leq j \leq n} (\mu + \alpha_i + \cdots + \alpha_j \mid \alpha_i + \cdots + \alpha_j) \\
&= 2 \sum_{1 \leq i \leq j \leq n-1} \left(\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) + \varepsilon_i - \varepsilon_{j+1} \mid \varepsilon_i - \varepsilon_{j+1} \right) \\
&\quad + 2 \sum_{1 \leq i \leq n} \left(\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) + \varepsilon_i \mid \varepsilon_i \right) \\
&= 2 \cdot \frac{n(n-1)}{2} \cdot 2 + 2 \cdot n \cdot \frac{3}{2} \\
&= n(2n - 2 + 3) = n(2n + 1)
\end{aligned}$$

In the calculation, we have used the fact that $\lambda - \mu = \alpha_1 + \cdots + \alpha_n$, so that the only positive roots α which can contribute are of the form $\alpha_i + \cdots + \alpha_j$, and they do so only for the first multiple ($m = 1$).

Now

$$\begin{aligned}
& ((\lambda + \rho \mid \lambda + \rho) - (\mu + \rho \mid \mu + \rho)) \\
&= (\lambda + \mu + 2\rho \mid \lambda - \mu) \\
&= ((2n + 1)\varepsilon_1 + (2n - 2)\varepsilon_2 + (2n - 4)\varepsilon_3 + \cdots + 2\varepsilon_n \mid \varepsilon_1) \\
&= 2n + 1.
\end{aligned}$$

Thus $\dim V(\lambda)_\mu = n$, and we are done. \square

In the construction of the Weyl module $V(\lambda)$, we have chosen a highest weight vector v . We will now give an explicit basis for $V(\lambda)_\mu$, in terms of the Chevalley generators f_i , $1 \leq i \leq n$.

Proposition 7.3. *A basis for $V(\lambda)_\mu$ is given by (v_1, \dots, v_n) , where*

$$v_i = f_i f_{i+1} \cdots f_n f_{i-1} \cdots f_1 v$$

Proof. The weight space $V(\lambda)_\mu$ is certainly generated by elements of the form $f_{i_1} \cdots f_{i_n} v$, where (i_1, \dots, i_n) is some permutation of $(1, \dots, n)$. Since there are no multiplicities, we do not have to worry about divided powers.

Now, in order to get a non-zero vector, we have to take i_n equal to 1 or n . If we choose $i_n = 1$, then we get $f_1 v$, a vector of weight $\varpi_2 + \varpi_n$, and for the next step we have to choose between 2 and n for i_{n-1} to get a non-zero result, and so on. Using the commutation relations, we can assume that we first apply f_1, f_2, \dots, f_{i-1} , and then f_n, f_{n-1}, \dots, f_i , for some i between 1 and n .

Thus $V(\lambda)_\mu$ is generated by (v_1, \dots, v_n) . By Lemma 7.2, we have $\dim V(\lambda)_\mu = n$, so this is a basis. By the way, we see that it is also a basis of $V_{\mathbb{Z}}(\lambda)_\mu$. \square

Alternatively, we can use the tableaux combinatorics, as explained in [KN94]. We can see $V(\lambda) = V(\varpi_1 + \varpi_n)$ as the submodule of $V(\varpi_1) \otimes V(\varpi_n)$ generated by a highest weight vector. The Weyl module $V(\varpi_1)$ is the natural representation

of dimension $2n + 1$, and $V(\varpi_n)$ is the spin representation, of dimension 2^n . With the notation of [KN94, §5], we have

$$v_i = \begin{array}{|c|c|} \hline 1 & i+1 \\ \hline 2 & \\ \hline \vdots & \\ \hline \widehat{i+1} & \\ \hline \vdots & \\ \hline n & \\ \hline i+1 & \\ \hline \end{array}$$

for $1 \leq i \leq n - 1$, and

$$v_n = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 2 & \\ \hline \vdots & \\ \hline n & \\ \hline \end{array}$$

Proposition 7.4. *The matrix of the contravariant form $(\cdot|\cdot)$ on $V_{\mathbb{Z}}(\lambda)_{\mu}$, in the basis (v_1, \dots, v_n) , is given by:*

$$\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 2 & 1 & 0 \\ \vdots & \ddots & 1 & 2 & 1 \\ 0 & \cdots & 0 & 1 & 3 \end{pmatrix}$$

The elementary divisors of this matrix are $(1, 1, \dots, 1, 2n + 1)$.

Proof. Using the commutation relations and the fact that v is a highest weight vector, we find:

$$\begin{aligned} (v_i | v_i) &= (v | e_1 \dots e_{i-1} e_n \dots e_i f_i \dots f_n f_{i-1} \dots f_1 v) \\ &= \prod_{j=i}^n \langle \lambda - \alpha_1 - \dots - \alpha_{i-1} - \alpha_{j+1} - \dots - \alpha_n, \alpha_j^{\vee} \rangle \\ &\quad \times \prod_{j=1}^{i-1} \langle \lambda - \alpha_1 - \dots - \alpha_{j-1}, \alpha_j^{\vee} \rangle \\ &= \begin{cases} 2 & \text{if } 1 \leq i \leq n - 1, \\ 3 & \text{if } i = n. \end{cases} \end{aligned}$$

Similarly, for $1 \leq i < j \leq n$, we have

$$\begin{aligned}
(v_i | v_j) &= (v | e_1 \dots e_{i-1} e_n \dots e_i f_j \dots f_n f_{j-1} \dots f_1 v) \\
&= \prod_{k=i+1}^{j-1} \langle \lambda - \alpha_1 - \dots - \alpha_{i-1}, \alpha_k^\vee \rangle \\
&\quad \times \prod_{k=j}^n \langle \lambda - \alpha_1 - \dots - \alpha_{i-1} - \alpha_{k+1} - \dots - \alpha_n, \alpha_k^\vee \rangle \\
&\quad \times \prod_{k=1}^i \langle \lambda - \alpha_1 - \dots - \alpha_{k-1}, \alpha_k^\vee \rangle \\
&= 0^{j-i-1} \times 1^{n+1-j} \times 1^i \\
&= \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This determines the matrix of the contravariant form (which is symmetric). Now, this matrix has the same elementary divisors as:

$$\begin{pmatrix} 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & 2 & 1 \\ 0 & \dots & \dots & 0 & 1 & 3 \\ 2 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

By induction, we can replace the last line by $(0, \dots, 0, i+1, i, 0, \dots, 0)$ without changing the elementary divisors (where the first non-zero entry is in column i), for i up to $n-1$. Then we can replace the last line by $(0, \dots, 0, 2n+1)$, hence the result. \square

Theorem 7.5. *Let $G = \text{Spin}_{2n+1}$, the simply-connected simple group of type B_n , and let $\lambda = \varpi_1 + \varpi_n$, $\mu = \varpi_n$ in the numbering of [Bou68]. Then we have $d_{\lambda\mu} = 1$ if ℓ divides $2n+1$, and 0 otherwise.*

Proof. This follows from the preceding results. \square

8. NON-EQUIVALENCES OF SINGULARITIES

Theorem 8.1. *(1) The singularities a_n and ac_n are not equivalent.
(2) The singularities a_2 , ac_2 and ag_2 are pairwise non-equivalent.
(3) The singularities c_2 and cg_2 are not equivalent.*

Proof. For each pair of singularities, we proceed as follows: if the two singularities were equivalent, then they would have the same intersection cohomology stalks, both in characteristic zero and in characteristic ℓ , and thus the corresponding decomposition numbers for perverse sheaves should be the same; but these are also decomposition numbers for a reductive group, and in each case, we see that the decomposition numbers differ for some primes ℓ .

The decomposition number for the singularity a_n is 1 if ℓ divides $n+1$, and 0 otherwise, whereas the decomposition number for the singularity ac_n is 1 if ℓ divides

$2n + 1$, and 0 otherwise. Moreover $n + 1$ and $2n + 1$ are coprime. Hence these two singularities are not equivalent.

The decomposition number for a_2 (resp. ac_2 , ag_2) is 1 for $\ell = 3$ (resp. $\ell = 5, 7$), and 0 otherwise (for the ac_2 and ag_2 cases, one can for example consult the tables in [Lüb01]). Hence these singularities are pairwise non-equivalent.

The decomposition number for c_2 and cg_2 is 1 if $\ell = 2$ (resp. $\ell = 3$) and 0 otherwise (again, one can use the tables in [Lüb01] for cg_2). Hence these two singularities are not equivalent. \square

Remark 8.2. We do not need the full strength of Theorem 7.5, if we just want to prove the non-equivalence of singularities for the ac_n case. If $\lambda \rightsquigarrow \mu$ is a minimal degeneration, by the strong linkage principle, if $d_{\lambda\mu} \neq 0$ then we must have

$$\mu = s_{\beta, m\ell}(\lambda) = \lambda - (\langle \lambda + \rho, \beta^\vee \rangle - m\ell)\beta$$

where $\beta = \lambda - \mu$, hence ℓ must divide $\langle \lambda + \rho, \beta^\vee \rangle - 1$. In the case of the singularity ac_n , β is the short dominant root, and we have

$$\begin{aligned} & \langle \lambda + \rho, \beta^\vee \rangle - 1 \\ &= \langle 2\varpi_1 + \varpi_2 + \cdots + \varpi_{n-1} + 2\varpi_n, 2\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee \rangle - 1 \\ &= 4 + 2(n-2) + 2 - 1 = 2n + 1. \end{aligned}$$

Besides, $n + 1$ and $2n + 1$ are coprime. Hence, choosing a prime number ℓ dividing $n + 1$, the decomposition number is one for a_n , and zero for ac_n , so the two singularities cannot be equivalent.

In the rank two case, choosing $\ell = 7$, the decomposition number is 1 for ag_2 and 0 for ac_2 (in this case $2n + 1 = 5$), which proves that they are also non-equivalent.

9. NON-SMOOTHNESS

We will now give a new proof of the fact that the smooth locus of the closure of an orbit in the affine Grassmannian is reduced to the orbit itself.

Representation-theoretic proof of Theorem 2.4. As in the proof of Malkin-Ostrik-Vybornov, we need only check that each minimal degeneration $\lambda \rightsquigarrow \mu$ is non-smooth. To this end, we will prove that there is some prime number ℓ for which it is not \mathbb{F}_ℓ -smooth. For this, it is enough to prove that there is some prime number ℓ for which the decomposition number $d_{\lambda\mu}$ for the corresponding perverse sheaves is non-trivial. By the geometric Satake isomorphism, this is a decomposition number for the reductive group G .

In case 1, we have $d_{\lambda\mu} = 1$ if ℓ divides $n + 1$ and 0 otherwise, thus the A_n singularity is not \mathbb{F}_ℓ -smooth for ℓ dividing $n + 1$.

In case 2, we have $d_{\lambda\mu} = \dim_{\mathbb{F}_\ell} \mathbb{F}_\ell \otimes_{\mathbb{Z}} P(\Phi_{I_{\text{sh}}})/Q(\Phi_{I_{\text{sh}}})$, thus this minimal singularity is not \mathbb{F}_ℓ -smooth for ℓ dividing $|P(\Phi_{I_{\text{sh}}})/Q(\Phi_{I_{\text{sh}}})|$.

In case 3, we have $d_{\lambda\mu} = 1$ if ℓ divides $2n + 1$ and 0 otherwise, thus the ac_n singularity is not \mathbb{F}_ℓ -smooth for ℓ dividing $2n + 1$.

In case 4, we have $d_{\lambda\mu} = 1$ if $\ell = 7$ and 0 otherwise, thus the ag_2 singularity is not \mathbb{F}_7 -smooth.

In case 5, we have $d_{\lambda\mu} = 1$ if $\ell = 3$ and 0 otherwise, thus the cg_2 singularity is not \mathbb{F}_3 -smooth.

So, in all cases, there is at least one prime number ℓ for which the decomposition number is non-trivial, so all the minimal degenerations are non-smooth, and the result follows. \square

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